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# TESTS FOR MONOTONE FAILURE RATE BASED ON NORMALIZED SPACINGS

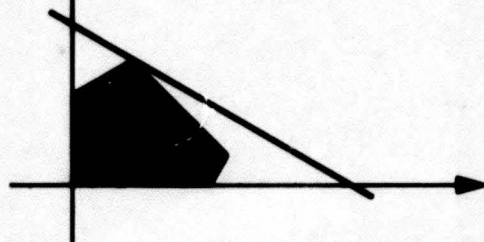
by

Peter J. Bickel and Kjell A. Doksum

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TESTS FOR MONOTONE FAILURE RATE  
BASED ON NORMALIZED SPACINGS

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# ABSTRACT

Let  $X_{(1)} < \dots < X_{(n)}$  be the order statistics of a random sample from a population with density  $f$  and distribution function  $F$  such that  $F(0) = 0$ . Let  $q(t) = f(t)(1 - F(t))^{-1}$  be the failure rate of  $F$ . In testing  $H_0: q(t) = \lambda$  vs.  $H_1: q(t) \neq \lambda$ , Proschan and Pyke (Vth Berk. Symp.) considered certain statistics based on  $R_1, \dots, R_n$ , the ranks of the normalized sample spacings  $D_i = (n - i + 1)(X_{(i)} - X_{(i-1)})$ ,  $1 \leq i \leq n$ ,  $X_{(0)} = 0$ . They show that these statistics are asymptotically normal for fixed  $F$  and compute the efficacy of one of these statistics for selected distributions. We show that asymptotic normality holds also for sequences of alternatives approaching  $H_0$  as  $n \rightarrow \infty$  and conclude that the above efficacies yield Pitman efficiencies. The statistic  $V = -\sum_{i=1}^n i R_i$  is asymptotically equivalent to the one considered by Proschan and Pyke. If  $S = -\sum_{i=1}^n i \log [1 - R_i(n+1)^{-1}]$  then the Pitman efficiency of  $V$  to  $S$  is  $3/4$ . If  $\{f_{\theta_n}\}$  is a sequence of alternative densities, let  $h(t) = [\partial \log f_{\theta}(t) / \partial \theta]_{\theta=0}$ ,  $c_1 = [\int_{a_1}^{\infty} h'(t) \exp(-t) dt] [1 - i(n+1)^{-1}]^{-1}$ ,  $a_1 = -\log \{1 - i(n+1)^{-1}\}$ . Then  $\sum_{i=1}^n c_i D_i$  is asymptotically most powerful for  $\{f_{\theta_n}\}$  if the scale parameter  $\lambda$  is known. However, the statistic  $S$  is nowhere asymptotically most powerful, although it is the asymptotically most powerful linear rank statistic for a suitable parametric family  $\{f_{\theta_n}\}$ . A comparative study of these and other statistics is given in terms of Pitman efficiencies, and Monte Carlo power.

#### ACKNOWLEDGEMENT

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# TESTS FOR MONOTONE FAILURE RATE BASED ON NORMALIZED SPACINGS

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## 1. Introduction and Summary

Let  $F$  be a distribution with density  $f$ , and let  $q(t) = f(t)/[1 - F(t)]$  be the failure rate of  $F$ . Tests for constant versus monotone increasing failure rate based on the ranks of the normalized spacings between the ordered observations have been considered by Proschan and Pyke [10]. They show that these statistics are asymptotically normally distributed for fixed alternatives  $F$  and compute the ratios of the efficacies of one of their rank tests to the best statistics for Weibull and Gamma alternatives.

In this paper, it is shown that asymptotic normality holds also for sequences of alternatives  $\{F_{\theta_n}\}$  that approach the  $H_0$  distribution  $1 - \exp(-\lambda t)$ ,  $t \geq 0$ , as  $n \rightarrow \infty$ ; and that the above mentioned ratios of efficacies are in fact Pitman efficiencies.

Let  $R_1, \dots, R_n$  be the ranks of the normalized spacing,  $T_1 = \sum 1 R_i$  and  $T_2 = -\sum 1 \log[1 - R_i/(n+1)]$ . Then  $T_1$  is asymptotically equivalent to the Proschan Pyke statistic. It is shown that the Pitman efficiency satisfies

$$e(T_1, T_2) \equiv 3/4 \quad (1.1)$$

for all sequences of alternatives  $\{F_{\theta_n}\}$  and thus  $T_1$  is asymptotically inadmissible.

Statistics that are linear in the normalized spacings and asymptotically most powerful for parametric alternatives  $\{F_{\theta_n}\}$ , if the scale parameter  $\lambda$  is known, are derived, and it is shown that the rank statistics that are

asymptotically most powerful in the class of linear rank tests, are nowhere most powerful in the class of all tests, when  $\lambda$  is known.

If  $\lambda$  is unknown, studentizing of the linear normalized spacing tests which are asymptotically most powerful for  $\lambda$  known lead to procedures which have only the same asymptotic power as the most powerful linear rank tests.

Unbiasedness is shown for tests that are monotone in the normalized spacings, and Monte Carlo power estimates are used to compare the various statistics with the likelihood ratio tests considered by Barlow [1].



## 2. Tests Monotone in the Normalized Spacings

Let  $X_1, \dots, X_n$  be a random sample from a population with a continuous distribution  $F$  satisfying  $F(0) = 0$ , and let  $0 = X_{(0)} < X_{(1)} < \dots < X_{(n)}$  be the order statistics. The normalized sample spacings  $D_1, \dots, D_n$  are defined by  $D_i = (n - i + 1)(X_{(i)} - X_{(i-1)})$ ,  $i = 1, \dots, n$ ; and  $R_i$  denotes the rank of  $D_i$  among  $D_1, \dots, D_n$ . The problem is to test  $H_0$ : " $-\log[1 - F(x)] = \lambda x$  on  $[0, \infty)$  for some positive constant  $\lambda$ " against  $H_1$ : " $-\log[1 - F(x)]$  is convex and not of the form  $\lambda x$  on  $[0, \infty)$ ." Note that the only distributions satisfying  $H_0$  are

$$K_\lambda(x) = 1 - e^{-\lambda x}; \quad x \geq 0, \lambda > 0. \quad (2.1)$$

Under  $H_0$  it is well known that  $(D_1, \dots, D_n)$  has the same distribution as a random sample from a population with distribution  $K_\lambda$ , while under the alternative there is a downward trend in the sense that  $P(D_i \geq D_j) < \frac{1}{2}$  whenever  $i > j$  (see [10]). One thus defines a test  $\phi = \phi(D_1, \dots, D_n)$  to be *monotone in the D's* if

$$\begin{aligned} \phi(D'_1, \dots, D'_n) &\leq \phi(D_1, \dots, D_n) \text{ for} \\ \text{all } (D_1, \dots, D_n) \text{ and } (D'_1, \dots, D'_n) \text{ such} \\ \text{that } i < j \text{ and } D'_i \geq D'_j \text{ implies } D_i \geq D_j. \end{aligned} \quad (2.2)$$

Following van Zwet [9], one defines a distribution  $F_1$  to have a more slowly increasing failure rate than  $F$ , written  $F_1 \geq_c F$ , if  $F_1^{-1} F$  is convex. Here  $F_1^{-1} F$  is defined by  $P(F_1^{-1} F(X) \leq x | F) = F_1(x)$ ,  $x \geq 0$ .

### Theorem 2.1:

*Monotone tests have monotone power, i.e., if  $\phi$  is a monotone test and if*

$F_1 \geq_c F$ , then

$$E(\phi \mid F_1) \leq E(\phi \mid F). \quad (2.3)$$

Proof:

Since  $F_1^{-1} F$  is increasing,  $X'_{(i)} = F_1^{-1} F(X_{(i)})$  is the  $i^{\text{th}}$  order statistic in a random sample from a population with distribution  $F_1$ . Let  $D'_i = (n - i + 1) (X'_{(i)} - X'_{(i-1)})$ ,  $i = 1, \dots, n$ . Since  $F_1^{-1} F$  is convex,  $i < j$  and  $D'_i \geq D'_j$  implies  $D_i \geq D_j$ . From (2.2) one obtains

$$\phi(D'_1, \dots, D'_n) \leq \phi(D_1, \dots, D_n), \quad (2.4)$$

and (2.3) follows upon taking expectations in (2.4).

Note that a test  $\phi$  is *similar* if  $E(\phi \mid K_\lambda)$  is independent of  $\lambda$ .

Thus all rank tests are similar.

Corollary 2.1:

*All monotone tests are similar and unbiased.*

Proof:

Similarity follows since (2.2) implies  $\phi(D_1, \dots, D_n) = \phi(\lambda D_1, \dots, \lambda D_n)$ .

Unbiasedness follows by letting  $F_1 = K_\lambda$  in Theorem 2.1.

Corollary 2.2:

*If  $-c_n(i)$  and  $J_n(i)$  are nondecreasing in  $i = 1, \dots, n$ , then the test that rejects when*

$$\sum_{i=1}^n c_n(i) J_n(R_i) \geq C_1$$



has monotone power and is unbiased.

Proof:

Using the notation of Theorem 2.1, define  $R'_i$  to be the rank of  $D'_i$  among  $D'_1, \dots, D'_n$ . Then  $i < j$  and  $R'_i \geq R'_j$  implies  $R_i \geq R_j$ . From Corollary 2 of Lehmann [9] one obtains

$$\sum_{i=1}^n c_n(i) J_n(R'_i) \leq \sum_{i=1}^n c_n(i) J_n(R_i). \quad (2.6)$$

It follows that the test is monotone, and Theorem 2.1 and Corollary 2.1 apply.

Remark 2.1:

The results of this section can be used to obtain bounds on the power of monotone tests. For instance, if  $\beta(V_n, F_2)$  denotes the power of the Proschan Pyke statistic  $V_n = \text{"number of pairs } (i, j) \text{ with } i < j \text{ and } D_i \geq D_j \text{"}$  for the Weibull distribution  $F_2$ , then the power satisfies  $\beta(V_n, F) \geq \beta(V_n, F_2)$  for all distributions  $F$  such that  $F_2^{-1}F$  is convex.  $\beta(V_n, F_2)$  is given in [1].

Remark 2.2:

Barlow and Proschan [13] have shown that if  $c_n(1) \geq \dots \geq c_n(n)$ , the test that rejects when

$$\frac{\sum_{i=1}^n c_n(i) D_i}{\sum_{i=1}^n D_i} \geq C_2$$

is similar and unbiased.

### 3. Asymptotically Most Powerful Tests

Let  $\{f_{\theta,\lambda} : \theta \geq 0, \lambda > 0\}$  be a class of densities such that  $f_{0,\lambda}(x) = \lambda \exp(-\lambda x)$ ,  $x \geq 0$ , and such that

$$h_\lambda(x) = \frac{\partial}{\partial \theta} \log f_{\theta,\lambda}(x) \Big|_{\theta=0} \quad (3.1)$$

exists.

We suppose  $\lambda$  is a scale parameter, i.e.,

$$P_{\theta,\lambda}(\lambda X \leq t) = P_{\theta,1}(X \leq t) . \quad (3.2)$$

For test " $\theta = 0$ " versus " $\theta > 0$ ", the locally most powerful test rejects " $\theta = 0$ " for large values of

$$T_n = n^{-1/2} \sum_{i=1}^n h_\lambda(x_i) . \quad (3.3)$$

We will consider sequences of alternatives  $\{f_{\theta_n,\lambda}\}$  such that

$$\lim_{n \rightarrow \infty} n^{1/2} \theta_n = b \text{ for some } 0 \leq b < \infty . \quad (3.4)$$

A sequence  $\{f_{\theta_n,\lambda}\}$  is said to be *contiguous* to  $f_{0,\lambda}$  (in the sense of LeCam - Hájek) if for any sequence of random variables  $R_n(X_1, \dots, X_n)$ ,  $R_n \rightarrow 0$  in  $P_0$  probability implies  $R_n \rightarrow 0$  in  $P_{\theta_n}$  probability, where  $P_{\theta}$  denotes the probability distribution of  $X_1, \dots, X_n$  if  $f_{\theta,\lambda}$  is true. The following condition implying contiguity for sequences as in (3.4) can be obtained from LeCam [8].

$$(a) \quad \partial f_{\theta,\lambda}(x) / \partial \theta \neq 0 \text{ whenever } f_{\theta,\lambda}(x) > 0 . \quad (3.5)$$

$$(b) \quad \text{For some } \delta > 0 \text{ and all } \theta \in [0, \delta], \quad 0 < H(\theta) = \int_0^\infty [\partial f_{\theta,\lambda}(x) / \partial \theta]^2 [f_{\theta,\lambda}(x)]^{-1} dx < \infty, \text{ and } H(\theta) \text{ is continuous in } \theta .$$

and  $H(\theta)$  is continuous in  $\theta$

$$(c) \lim_{|h| \rightarrow 0} \int_0^\infty \left\{ h^{-1} \left[ f_{\theta+h, \lambda}^{1/2}(x) - f_{\theta, \lambda}^{1/2}(x) \right] - \frac{1}{2} \left[ \partial f_{\theta, \lambda}(x) / \partial \theta \right] f_{\theta, \lambda}^{-1/2}(x) \right\}^2 dx = 0 \text{ for } \theta \in [0, \delta] .$$

An easy sufficient condition implying (3.5) (b) and (c) is

$$\int_0^\infty \sup \left\{ \left[ \frac{\partial f_{\theta, \lambda}(x)}{\partial \theta} \right]^2 [f_{\theta, \lambda}(x)]^{-1} : 0 \leq \theta \leq \delta \right\} dx < \infty \quad (3.6)$$

for some  $\delta > 0$ .

It also follows from LeCam's work that under condition (3.5) we have

$$\begin{aligned} & \left[ b n^{-1/2} \sum_{i=1}^n h_{\lambda}(X_i) - \frac{b^2}{2} E_{\theta, \lambda}(h_{\lambda}^2(X_1)) \right] \\ & - \sum_{i=1}^n \left\{ \log f_{\theta_n, \lambda}(X_i) - \log f_{\theta, \lambda}(X_i) \right\} \rightarrow 0 \\ & \text{in } P_{\theta} \text{ and hence in } P_{\theta_n} \text{ probability.} \end{aligned} \quad (3.7)$$

According to Wald [12], a sequence of level  $\alpha$  tests  $\{\phi_n\}$  is said to be asymptotically most powerful ( $\lambda$  known) if

$$\lim_n \sup \{ E_{\theta, \lambda}(\psi_n) - E_{\theta, \lambda}(\phi_n) : \theta > 0, E_{\theta, \lambda}(\psi_n) \leq \alpha \} = \alpha .$$

Using contiguity we can prove Wald's [12] main theorem under weaker conditions.

### Theorem 3.1:

Suppose (3.5) holds and,

$$n^{-1/2} \sum_{i=1}^n h_{\lambda}(X_i) \rightarrow \infty \quad (3.9)$$

in  $P_{\theta_n, \lambda}$  probability if  $n^{1/2} \theta_n \rightarrow \infty$ . Then the sequence of level  $\alpha$  tests

$\phi_n$  which reject for suitably large values of  $\sum_{i=1}^n h_\lambda(X_i)$  is asymptotically most powerful.

**Proof:**

The result is an easy consequence of the Neyman - Pearson Lemma, (3.7) and (3.9) . ||

It is easy to see that the  $\phi_n$  are locally most powerful.

Our next result employs Theorem 4.1 of the following section. We need the notation

$$\sigma^2(h_1) = \int_0^\infty h_1^2(x) e^{-x} dx, \text{ and} \quad (3.10)$$

$$S_n = n^{-\frac{1}{2}} \sum_{i=1}^n a\left(\frac{i}{n+1}\right) (D_i - \lambda^{-1}), \text{ where} \quad (3.11)$$

$$a(u) = (1-u)^{-1} \int_{-\log(1-u)}^\infty h_1'(x) e^{-x} dx, \quad 0 < u < 1. \quad (3.12)$$

**Theorem 3.2:**

If (3.4) and (3.5) hold, and if  $h_\lambda$  of (3.1) satisfies the conditions of Theorem 4.1, then  $\lambda S_n \sigma^{-1}(h_1)$  has under  $H_0$  asymptotically a standard normal distribution. Moreover, the test that rejects for large values of  $S_n$  is asymptotically most powerful.

**Proof:**

Since  $\lambda$  is scale parameter, and since  $S_n$  is linear, one obtains  $P_{\theta, \lambda}(\lambda S_n \leq x) = P_{\theta, 1}(S_n \leq x)$ . Thus one may assume without loss of generality that  $\lambda = 1$ . Since  $\sigma^2(h_1)$  is the asymptotic variance of  $T_n$ , the asymptotic normality follows at once from Theorem 4.1. By Theorem 4.1, the test rejecting for large values of  $T_n$  is asymptotically most powerful. Thus

it is enough to show that  $T_n - S_n \rightarrow 0$  in  $P_{\theta_n}$  probability. However, this follows at once from the contiguity condition (3.5) and Theorem 4.1. ||

Remark:

Theorem 3.2 and the contiguity condition implies the asymptotic normality of  $S_n$  also under sequences of alternatives for which (3.4) is satisfied, i.e., if

$$\mu_n(h_1) = n^{1/2} \int_0^\infty h_1(x) f_{\theta,1}(x) dx, \quad (3.13)$$

then  $[\lambda S_n - \mu_n(h_1)]/\sigma(h_1)$  has asymptotically a standard normal distribution.

#### 4. Linear Approximations to Locally and Asymptotically most Powerful Statistics

As we have seen, locally and asymptotically most powerful test statistics for parametric alternatives are of the form

$$T_n = n^{-1/2} \sum_{i=1}^n h(X_i) = n^{-1/2} \sum_{i=1}^n h(X_{(i)})$$

for some function  $h$  on  $[0, \infty)$ . In order to compute asymptotic efficiencies, we will need the following result which gives conditions under which  $T_n$  can be approximated by a statistic linear in the  $D_i$ 's, and given by,

$$S_n = n^{-1/2} \sum_{i=1}^n a\left(\frac{i}{n+1}\right) (D_i - 1) \quad (4.2)$$

where,

$$a(u) = (1 - u)^{-1} \int_{-\log(1-u)}^{\infty} h'(x) e^{-x} dx, \quad 0 < u < 1.$$

#### Theorem 4.1:

Let  $h$  be any function such that

(i)  $h'$  is continuous on  $(0, \infty)$ ,

(ii)  $\int_0^{\infty} h(t) e^{-t} dt = 0$ ,  $0 < \int_0^{\infty} h^2(t) e^{-t} dt < \infty$ ,

(iii) either one of the following holds

(a)  $h'(-\log(1 - u))$ ,  $0 < u < 1$ , satisfies assumption E of Chernoff, Gastwirth and John. (1967) and



$$\int_0^{\infty} |h'(v)| e^{-\frac{v}{2}} (1 - e^{-v})^{\frac{1}{2}} dv < \infty.$$

(b)  $h'(t)$  changes sign only a finite number of times as  $t \rightarrow 0$  or  $\infty$  and  $h'$  does not vanish infinitely often.

Moreover, suppose that the  $X$ 's have the exponential density  $\exp(-x)$ ,  $x \geq 0$ .

Then  $T_n - S_n$  tends to zero in probability.

The proof is deferred to the appendix; Section 7.

Remark:

It may also be shown that if (i), (ii) and (iii) (b) hold and  $\lambda = 1$ , then  $E_{0,1}(T_n - S_n)^2 \rightarrow 0$ .



### 5. Asymptotic Normality and Inefficiency of Rank Statistics

Let Pitman asymptotic efficiency be defined as usual (e.g., Hodges and Lehmann [6]). In this section we shall show the asymptotic normality of statistics of the form

$$W_n = n^{-1/2} \sum_{i=1}^n (c_i - \bar{c}) J\left(\frac{R_i}{n+1}\right), \quad \bar{c} = n^{-1} \sum_{i=1}^n c_i, \quad (5.1)$$

and compute their Pitman efficiencies with respect to the asymptotically most powerful statistics of Section 4. Let  $h_\lambda$  be as defined in (3.1) and let

$$a_1 = a\left(\frac{1}{n+1}\right) \quad \text{where} \quad a(u) = (1-u)^{-1} \int_{-\log(1-u)}^{\infty} h_1'(x) e^{-x} dx. \quad \text{Then} \quad (5.2)$$

$$S_n = n^{-1/2} \sum_{i=1}^n a_1 (D_i - \lambda^{-1}) \quad (5.3)$$

is the asymptotically most powerful statistic of Section 3.

#### Theorem 5.1:

If (3.4) and (3.5) hold, if  $h_\lambda$  satisfies the conditions of Theorem 4.1 of this paper, and if  $\{c_i\}$  and  $\left\{J\left(\frac{1}{n+1}\right)\right\}$  satisfy the conditions of Theorem 4.1 of Hájek [5], then for the sequence of alternatives given in (3.4)

$[W_n - \mu(W_n)]/\sigma(W_n)$  has asymptotically a standard normal distribution, where

$$\mu(W_n) = bn^{1/2} \left[ \frac{1}{n} \sum_{i=1}^n a_1 (c_i - \bar{c}) \right] \int_0^1 J(u) [-\log(1-u) - 1] du, \quad \text{and} \quad (5.4)$$

$$\sigma^2(W_n) = \left[ \frac{1}{n} \sum_{i=1}^n (c_i - \bar{c})^2 \right] \left[ \int_0^1 J^2(u) du - \left( \int_0^1 J(u) du \right)^2 \right] \quad (5.5)$$

Proof:

Since the ranks are scale invariant, assume without loss of generality that  $\lambda = 1$ . The results of Hajek [5] and the contiguity condition imply that

$$W_n - Q_n \rightarrow 0 \text{ in } P_{\theta_n, 1} \text{ probability where,} \quad (5.6)$$

$$Q_n = n^{-1/2} \sum_1^n (c_1 - \bar{c}) J(1 - \exp(-\lambda D_1)) . \quad (5.7)$$

It follows from the Lindeberg-Feller Theorem that under  $H_0$ , the joint limiting distribution of  $bS_n/\sigma_n(bS_n)$  and  $Q_n/\sigma(W_n)$  is the bivariate normal distribution with means zero, variances one, and covariance

$$b \left[ n^{-1} \sum_1^n a_1 (c_1 - \bar{c}) \right] \left[ \int_0^1 J(u) [-\log(1 - u) - 1] du \right]$$

$$\text{where } \sigma_n^2(bS_n) = b^2 n^{-1} \sum_1^n a_1^2 .$$

The result now follows from LeCam's third lemma (see Hájek and Sidák ([4] p.208)).

For each vector  $c = (c_1, \dots, c_n)$ , define

$$V_n(c) = \frac{1}{n} \sum_1^n (c_1 - \bar{c})^2 \quad (5.8)$$

and for two vectors  $c$  and  $a$  define

$$\text{Cor}_n^2(a, c) = \frac{\left[ \frac{1}{n} \sum_1^n a_1 (c_1 - \bar{c}) \right]^2}{V_n(a) V_n(c)} . \quad (5.9)$$

Then Theorem 5.1 yields

Corollary 5.1:

Under the conditions of Theorem 5.1, the Pitman asymptotic relative efficiency of  $W_n$  to  $S_n$  is

$$e(W, S) = \text{Cor}^2(J(U), -\log(1 - U)) \lim_{n \rightarrow \infty} \left[ \text{Cor}_n^2(a, c) \frac{V_n(a)}{\frac{1}{n} \sum_{i=1}^n a_i^2} \right] \quad (5.10)$$

where  $U$  is an uniform  $(U(0, 1))$  random variable.

It is clear from (5.9) that one obtains the most efficient linear rank statistic  $W_n$  by taking  $J(u) = -\log(1 - u)$  and  $c_i = a_i$ . Note that the choice of  $J$  is independent of the alternative densities  $\{f_{\theta, \lambda}\}$ . Moreover,  $e(W, S)$  is less than one when

$$\lim(\bar{a})^2 = \left[ \int_0^\infty x h_1(x) e^{-x} dx \right]^2 = \text{Cor}^2(X_1, h_1(X_1)) \neq 0. \quad (5.11)$$

This is equivalent to having  $\bar{X}_n = n^{-1} \sum_{i=1}^n X_i$  correlated with the locally most powerful statistic  $T_n$ . It will be shown in the proof of the next result that  $\text{Cor}(\bar{X}_n, T_n) < 0$  when the failure rate is increasing. Let

$$q_\lambda(x, \theta) = f_{\theta, \lambda}(x) [1 - F_{\theta, \lambda}(x)]^{-1}, \quad x \geq 0, \quad (5.12)$$

denote the failure rate of  $F_{\theta, \lambda}$ . We assume in what follows that  $\partial q_\lambda(x, \theta) / \partial \theta$  exists and is continuous in  $(x, \theta)$  for  $0 \leq \theta \leq \delta$ ,  $x > 0$ , some  $\delta > 0$ . Let

$$L(x) = \partial q_\lambda(x, \theta) / \partial \theta \big|_{\theta = 0}. \quad (5.13)$$

Increasing failure rate clearly implies

$$L(x) \geq 0 \quad (5.14)$$

for all  $x > 0$ . We have

Theorem 5.2:

Suppose the conditions of Theorem 5.1 hold, that (5.14) is satisfied with strict inequality for  $x$  in some set of positive measure and that,

$$\int_0^{\infty} t e^{-t} L(t) dt < \infty. \quad (5.15)$$

Then each linear rank statistic of the form (5.1) is inefficient, i.e.,

$$e(W, S) < 1. \quad (5.16)$$

Proof:

Without loss of generality let  $\lambda = 1$ . Now,

$$f_{\theta,1}(x) = q_1(x, \theta) \exp \left\{ - \int_0^x q_1(t, \theta) dt \right\}. \quad (5.17)$$

Since  $\partial q_1(x, \theta)/\partial \theta$  is continuous in  $(x, \theta)$  we have differentiating under the integral sign,

$$h_1(x) = L(x) - \int_0^x L(t) dt. \quad (5.18)$$

Using Fubini's theorem and (5.15) we get if  $\lambda = 1$ ,

$$\text{cov}(X_1, h_1(X_1)) = \int_0^{\infty} x h(x) e^{-x} dx = - \int_0^{\infty} e^{-x} L(x) dx < 0 \quad (5.19)$$

by our assumptions. ||

Example 5.1:

An alternative for which there is an efficient rank test is provided by

$$f_{\theta,1}(x) = (1 - \theta)^{-1} [1 + \theta(-2x + \frac{1}{2}x^2)] e^{-x}, \theta < \frac{1}{2}. \quad (5.20)$$

For this density, the failure rate is not monotone. It is easy to see that  $\text{Cor}(X_1, h(X_1)) = 0$ . The efficient rank test rejects for large values of

$$n^{-\frac{1}{2}} \sum_{i=1}^n \left[ \log \left( 1 - \frac{i}{n+1} \right) + 1 \right] \left[ -\log \left( 1 - \frac{R_i}{n+1} \right) \right] \quad (5.21)$$

We conjecture that under the hypotheses of our theorem there is no sequence of rank tests which is asymptotically most powerful in the sense of Wald. Our best approximation to this result is Theorem 5.3.<sup>†</sup>

Definition:

We say a sequence of statistics  $M_n(X_1, \dots, X_n)$  is *efficient* if

$$\text{as. cor.} \left( M_n, \sum_{i=1}^n \left\{ \log f_{\theta_n, \lambda}(X_i) - \log f_{0, \lambda}(X_i) \right\} \right) = 1 \quad (5.22)$$

for every sequence of alternatives  $\theta_n$  satisfying (3.4). By as. cor. we refer to the asymptotic correlation computed under  $H_0$ .

It is easy to see that if (4.15) holds and

$$R_n \rightarrow \infty \quad (5.23)$$

in  $P_{\theta_n}$  probability if  $n^{\frac{1}{2}} \theta_n \rightarrow \infty$ , then an efficient sequence of  $M_n$ 's can be used to construct asymptotically most powerful tests for every level  $\alpha$ .

<sup>†</sup>E. Torgersen, using general considerations, has proved that our conjecture is implied by Theorem 5.3. A more direct proof using the special structure of this problem may also be given.

Theorem 5.3:

Under the assumption of Theorem 5.2 no efficient sequence of rank statistics exists.

We need an elementary lemma.

Lemma 5.4:

Let  $U_n$  be a sequence of random variables such that  $U_n \xrightarrow{P} 0$ .

Denote by  $U_n^{(k)}$  the random variable equal to  $U_n$  if  $|U_n| \leq k$  and equal to  $k \operatorname{sgn} U_n$  otherwise. Then there exists  $k_n \uparrow \infty$ , such that,

$$E \left[ U_n^{(k_n)} \right]^2 \rightarrow 0. \quad (5.24)$$

Proof:

Let  $k_n$  be the largest  $k$  such that  $E \left[ U_n^{(k)} \right]^2 \leq \frac{1}{n}$ .

Clearly  $k_n \uparrow \infty$ . ||

Proof of Theorem:

Suppose  $M_n(R_1, \dots, R_n)$  is an efficient sequence of rank statistics. Let  $\theta_n = n^{-1/2}$ ,  $\lambda = 1$ . Without loss of generality suppose,

$$M_n - \sum_{i=1}^n \left[ \log f_{\theta_n, 1}(X_i) - \log f_{0, 1}(X_i) \right] - \frac{1}{2} E_{0, 1}(h_1^2(X_1)) \rightarrow 0 \quad (5.25)$$

in  $P_{\theta_n}$  probability. By (3.7) and (3.12)

$$M_n - S_n \rightarrow 0 \quad (5.26)$$

in  $P_0$  probability. By the lemma there exist  $k_n$  such that

$$E_{0, 1} \left[ (M_n - S_n)^{(k_n)} \right]^2 \rightarrow 0. \quad (5.27)$$

But

$$E_{0,1} \left[ (M_n - S_n)^{(2k_n)} \right]^2 \geq E_{0,1} M_n^{(k_n)} - S_n^{(k_n)} \quad (5.28)$$

and

$$E_{0,1} \left( S_n - S_n^{(k_n)} \right)^2 \rightarrow 0. \quad (5.29)$$

From (5.27) - (5.29) we conclude that

$$E_{0,1} (S_n - E_{0,1}(S_n \mid R_1, \dots, R_n))^2 \rightarrow 0. \quad (5.30)$$

But,

$$U_n = E_{0,1}(S_n \mid R_1, \dots, R_n) = n^{-1/2} \int a\left(\frac{1}{n+1}\right) E_{0,1}(X_{(R_1)}) \quad (5.31)$$

is a statistic of the form (5.1). Theorem 5.2 and (5.30) are therefore incompatible. ||

We conjecture that rank tests which reject for large values of  $E_{0,1}(S_n \mid R_1, \dots, R_n)$  are asymptotically most powerful among all rank tests for the given model whatever be  $\lambda$ .

It is not difficult to see by using the method of Hoeffding [7] that the locally most powerful rank test rejects the null hypothesis for large values of the statistic

$$\begin{aligned} W_n &= E_{0,1}(T_n \mid R_1, \dots, R_n) \\ &= n^{-1/2} \sum_{i=1}^n E_{0,1} \left[ h_1 \left( \sum_{j=1}^i X_{(R_j)} (n-j+1)^{-1} \right) \right]. \end{aligned} \quad (5.32)$$

By the remark following Theorem 4.1 since

$$E_{0,1}(U_n - W_n)^2 \leq E_{0,1}(S_n - T_n)^2 \quad (5.33)$$



$U_n$  and  $W_n$  are asymptotically equivalent. This lends further support to our conjecture.

These rank statistics are of course usable even if  $\lambda$  is unknown, the situation which primarily concerns us, while the optimal statistics of Section 4 depend on  $\lambda$  and do not lead to similar tests. If we use the method of Barlow [1] and Nadler and Eilbott [14] and consider studentized statistics of the form

$$S_n^* = \frac{n^{\frac{1}{2}} \sum_{i=1}^n a\left(\frac{1}{n+1}\right) D_i}{\sum_{i=1}^n D_i} - n^{\frac{1}{2}} \sum_{i=1}^n a\left(\frac{1}{n+1}\right), \quad (5.34)$$

it is not difficult to show that for any fixed  $\lambda$  under the null hypothesis,

$$S_n^* - n^{-\frac{1}{2}} \lambda \sum_{i=1}^n \left( a\left(\frac{1}{n+1}\right) - \bar{a} \right) (D_i - \lambda^{-1}) \rightarrow 0 \quad (5.35)$$

in probability and hence by (5.6) the best studentized test of this form is asymptotically equivalent under contiguous alternatives to the rank test based on  $M_n(R_1, \dots, R_n)$  where,

$$M_n(R_1, \dots, R_n) = - \sum_{i=1}^n a\left(\frac{1}{n+1}\right) \log\left(1 - \frac{R_i}{n+1}\right). \quad (5.36)$$

We conjecture that the asymptotically most powerful linear rank tests, asymptotically equivalent to the studentized asymptotically most powerful linear spacings tests, are in fact asymptotically most powerful among all level  $\alpha$  tests which are similar for the hypothesis  $H_0$ ,  $\lambda$  unknown.

Finally we remark that the rank statistic whose Monte Carlo power is studied

in Section 6 is not  $U_n$  but an asymptotically equivalent simpler form given by,

$$U_n^* = n^{-1/2} \sum_{i=1}^n -L\left(-\log 1 - \left(\frac{i}{n+1}\right)\right) - \log\left(1 - \frac{R_i}{n+1}\right).$$

suggested by the discussion following Corollary 5.1.

## 6. Applications of the Theory and Monte Carlo Results

The results of the previous sections will now be applied to specific alternatives and specific statistics. The weights  $c_i$  of (5.1) will be of the form

$$c_i = c\left(\frac{i}{n+1}\right) \text{ for some function } c \text{ on } (0, 1) \quad (6.1)$$

and the efficiency (5.10) will be

$$e(W, S) = \text{Cor}^2(J(U), -\log(1 - U)) \text{Cor}^2(a(U), c(U)) \frac{\text{Var}(a(U))}{E(a^2(U))} \quad (6.2)$$

where  $U$  is an uniform  $(U(0, 1))$  random variable.

The statistics to be considered are:

$$W_0 = \sum_{i=1}^n -\left(\frac{i}{n+1}\right) \left(\frac{R_i}{n+1}\right)$$

$$W_1 = \sum_{i=1}^n -\left(\frac{i}{n+1}\right) \left[-\log\left(1 - \frac{R_i}{n+1}\right)\right]$$

$$W_2 = \sum_{i=1}^n \left[-\log 1 - \left(\frac{i}{n+1}\right)\right] \left[-\log\left(1 - \frac{R_i}{n+1}\right)\right]$$

$$W_3 = \sum_{i=1}^n -\left\{\log\left[-\log 1 - \left(\frac{i}{n+1}\right)\right]\right\} \left[-\log\left(1 - \frac{R_i}{n+1}\right)\right]$$

$$W_4 = \sum_{i=1}^n g\left(\frac{i}{n+1}\right) \left[-\log\left(1 - \frac{R_i}{n+1}\right)\right]$$

$$S_1 = \sum_{i=1}^n -\left(\frac{i}{n+1}\right) D_i$$

$$S_2 = \sum_{i=1}^n \left[\log\left(1 - \frac{i}{n+1}\right)\right] D_i$$

$$S_3 = \sum_{i=1}^n -\left\{ \log \left[ -\log \left( 1 - \frac{1}{n+1} \right) \right] \right\} D_i$$

$$S_4 = \sum_{i=1}^n g\left(\frac{1}{n+1}\right) \left[ -\log \left( 1 - \frac{R_i}{n+1} \right) \right]$$

$$\text{where } g(t) = (1-t)^{-1} \int_{-\log(1-t)}^{\infty} x^{-1} e^{-x} dx .$$

Large values are significant.

The alternative densities to be considered are listed below for  $\lambda = 1$  ;  
to obtain the general form, make the transformation  $f(x) \rightarrow \lambda f(\lambda x)$  .

$$f_{\theta}^{(1)}(x) = [1 + \theta(1 - e^{-x})] \exp \{ -[x + \theta(x + e^{-x} - 1)] \} \quad (\text{Makeham})$$

$$f_{\theta}^{(2)}(x) = (1 + \theta x) \exp \{ -(x + \frac{1}{2}\theta x^2) \} \quad (\text{linear F.R.})$$

$$f_{\theta}^{(3)}(x) = (1 + \theta) x^{\theta} \exp \{ -x^{(1+\theta)} \} \quad (\text{Weibull})$$

$$f_{\theta}^{(4)}(x) = [x^{\theta} e^{-x}] / \Gamma(1 + \theta) \quad (\text{Gamma})$$

for each density,  $x \geq 0$ ,  $\theta \geq 0$  ; and the null hypothesis is obtained for  $\theta = 0$  . Each of these densities have increasing failure rates for  $\theta > 0$  .  
Note that  $f_{\theta}^{(2)}$  has the linear F.R. (failure rate)  $1 + \theta x$  , while  $f_{\theta}^{(1)}$  has the failure rate  $1 + \theta(1 - e^{-x})$  .

From Theorem 3.1 it follows that  $S_1$  is asymptotically most powerful for  $f_{\theta}^{(1)}$  when  $\lambda$  is known,  $i = 1, 2, 3, 4$  . Theorem 5.1 implies that  $W_1$  is asymptotically most powerful for  $f_{\theta}^{(1)}$  in the class of linear rank statistics,  $i = 1, 2, 3, 4$  .

$W_0$  is asymptotically equivalent to the Proschan-Pyke statistic  $V_n$  (see Remark 3.1). It is uniformly improved asymptotically by  $W_1$  . We now list the efficiencies of  $W_1$  to the asymptotically most powerful statistic of Section 3.

The efficiencies are given in general as functions of  $h(x) = h_1(x) = [\partial \log f_{\theta,1}(x)/\partial \theta]_{\theta=0}$ . They are always independent of  $\lambda$ .

$$e(W_0) = \frac{9 \left[ \int_0^\infty h'(x) e^{-x} (\frac{1}{2}x - 1 + e^{-x}) dx \right]^2}{\int_0^\infty h^2(x) e^{-x} dx}$$

$$= \frac{9 \left[ \int_0^\infty h(x) e^{-x} (\frac{1}{2}x - 3/2 + 2e^{-x}) dx \right]^2}{\int_0^\infty h^2(x) e^{-x} dx}$$

$$e(W_1) = \frac{4}{3} e(W_0)$$

$$e(W_2) = \frac{\left[ \int_0^\infty h^1(x) e^{-x} (\frac{1}{2}x^2 - x) dx \right]^2}{\int_0^\infty h^2(x) e^{-x} dx}$$

$$e(W_3) = \frac{\left[ \int_0^\infty h^1(x) e^{-x} x(\log x + \gamma - 1) dx \right]^2}{\left( \frac{\pi^2}{6} - 2\gamma + 1 \right) \int_0^\infty h^2(x) e^{-x} dx}$$

$$e(W_4) = \frac{\int_0^\infty x^{-1} e^{-x} \int_0^{1-e^{-x}} (1-t)^{-1} (a(t) - \bar{a}) dt dx}{\left( \frac{\pi^2}{6} - 1 \right) \int_0^\infty h^2(x) e^{-x} dx}$$

where  $\gamma = .5772$  is Euler's constant,

$$a(t) = (1-t)^{-1} \int_{-\log(1-t)}^\infty h'(t) e^{-t} dt, \text{ and } \bar{a} = \int_0^1 a(t) dt.$$

As remarked in Section 5, if

$$S_i^* = S_i / n^{-1} \sum_{j=1}^n D_j - n^{\frac{1}{2}} \sum_{j=1}^n a\left(\frac{j}{n+1}\right), \quad i = 1, 2, 3, 4 \quad (6.3)$$

are the studentized linear spacings statistics, then  $S_i^*$  have the same efficiency as  $W_i$ , i.e.,  $e(S_i^*) = e(W_i)$   $i = 1, 2, 3, 4$ . These efficiencies are given in Table 6.1.

Tables 6.2 deal with a Monte Carlo study of the powers of the  $W_i$  and  $S_i^*$  statistics for the linear F.R., Weibull and Gamma distributions.

From the tables, the following conclusions are apparent: 1) The rank tests are uniformly less powerful than the corresponding studentized linear spacings tests. 2) Of all the tests considered in the above tables, the total time on test statistic  $S_1^*$  is generally best both on the basis of asymptotic efficiency and Monte Carlo power. 3) Barlow [1] has shown that for Weibull and Gamma alternatives,  $S_1^*$  is much better than the IFR likelihood ratio test and slightly worse than the IFRA likelihood ratio test.

Table 6.2 also gives the Monte Carlo power of the test which rejects for large values of

$$T_3 = \frac{\sum_{i=1}^n (1 - x_i) \log x_i}{\sum x_i}.$$

Using the results of Section 3, we see that this test is the studentized version of the locally most powerful test for the Weibull distribution. The table shows the nonrobustness of this test.

The last row in Table 6.2 gives the factor by which the efficiencies have to be multiplied in order to obtain the efficiencies with respect to the best linear rank tests (or the best studentized linear spacings tests).

TABLE 6.1  $e(W_i) = e(S_i^*)$

Statistic $f_{\theta}^{(i)}$	Makeham	Linear F.R.	Weibull	Gamma
$W_0$	$\frac{3}{16} = .1875$	$\frac{9}{32} = .2813$	$\frac{9(\log 2)^2}{4\gamma_0} = .5927$	$\frac{54(\log 2 - \frac{1}{2})^2}{\pi^2} = .2041$
$W_1$	$\frac{1}{4} = .25$	$\frac{3}{8} = .375$	$\frac{3(\log 2)^2}{\gamma_0} = .7903$	$\frac{72}{\pi^2}(\log 2 - \frac{1}{2})^2 = .2721$
$W_2$	$\frac{3}{16} = .1875$	$\frac{1}{2} = .50$	$\frac{1}{\gamma_0} = .5483$	$\frac{3}{2\pi^2} = .1520$
$W_3$	$\frac{3(\log 2)^2}{4(\gamma_0 - \gamma^2)} = .2416$	$\frac{1}{2(\gamma_0 - \gamma^2)} = .3355$	$\frac{1 - \gamma^2}{\gamma_0} = .8173$	$\frac{6\gamma^2(\pi^2 - 6)}{\pi^4 \gamma_0} = .0435$
$W_4$	$\frac{18(\log 2 - \frac{1}{2})^2}{\pi^2 - 6} = .1735$	$\frac{3}{4(\pi^2 - 6)} = .1938$	$\frac{6\gamma^2(\gamma_0 - \gamma^2)}{\pi^2 \gamma_0} = .0908$	$1 - \frac{6}{\pi^2} = .3921$
Rank efficiency factor	4	2	$\frac{\gamma_0}{\gamma_0 - \gamma^2} = 1.2235$	$\frac{\pi^2}{2} = 2.5505$

$$\gamma = .5772 \dots = \text{Euler's constant}, \gamma_0 = (\gamma - 1)^2 + \frac{\pi^2}{6} = 1.8237$$



TABLE 6.2: Monte Carlo power based 2,000 trials for samples of size 10 from Linear F. R., Weibull and Gamma distributions for significance level  $\alpha = .01, .05$  and  $.10$ .

LINEAR F. R.									
$\alpha$	.01	.05	.10	.01	.05	.10	.01	.05	.10
$\theta$	.50			1.00			2.50		

## WEIBULL

$\alpha$	.01	.05	.10	.01	.05	.10	.01	.05	.10	.01	.05	.10	.01	.05	.10	.01	.05	.10						
$\theta$	1.25			1.50			1.75			2.00			2.50			3.00			3.50			4.50		
$W_1$	.029	.136	.251	.073	.280	.477	.130	.420	.621	.217	.562	.757	.318	.727	.880	.405	.805	.932	.482	.832	.943	.538	.863	.954
$W_2$	.021	.112	.227	.055	.255	.426	.106	.340	.538	.174	.477	.670	.252	.605	.791	.317	.675	.846	.369	.731	.869	.403	.774	.895
$W_3$	.023	.150	.252	.059	.318	.506	.107	.470	.672	.188	.617	.800	.286	.797	.924	.388	.879	.963	.470	.702	.971	.529	.827	.985
$S_1^*$	.039	.149	.269	.112	.337	.507	.238	.546	.719	.400	.734	.869	.721	.941	.982	.905	.994	.999	.977	.999	1.00	1.00	1.00	1.00
$S_2^*$	.025	.146	.226	.079	.306	.448	.168	.513	.657	.310	.686	.813	.621	.916	.967	.837	.989	.997	.946	.998	1.00	.999	1.00	1.00
$S_3^*$	.043	.161	.277	.124	.352	.533	.270	.566	.736	.434	.754	.883	.740	.945	.986	.909	.992	.999	.977	.999	1.00	.999	1.00	1.00
$T_3$	.053	.178	.288	.146	.355	.501	.272	.526	.658	.425	.694	.816	.694	.881	.935	.868	.969	.983	.946	.993	.999	.994	1.00	1.00

## GAMMA

$\alpha$	.01 .05 .10	.01 .05 .10	.01 .05 .10	.01 .05 .10	.01 .05 .10	.01 .05 .10	.01 .05 .10	.01 .05 .10	.01 .05 .10
$\theta$	1.50	2.00	2.50	3.00	3.50	4.00	4.50	5.50	
$\lambda_1$	.036 .162 .288	.068 .275 .455	.110 .375 .583	.159 .447 .665	.195 .553 .733	.214 .576 .775	.232 .603 .797	.264 .628 .810	
$\lambda_2$	.036 .144 .249	.055 .225 .386	.084 .287 .482	.116 .356 .555	.142 .410 .600	.157 .499 .635	.158 .454 .664	.184 .498 .678	
$\lambda_3$	.023 .178 .312	.055 .323 .497	.093 .452 .659	.138 .548 .747	.176 .635 .814	.204 .691 .876	.221 .720 .879	.258 .746 .896	
$\lambda_1^*$	.045 .175 .298	.096 .328 .492	.203 .502 .680	.321 .647 .809	.434 .775 .893	.541 .864 .953	.651 .905 .960	.803 .971 .993	
$\lambda_2^*$	.033 .161 .251	.063 .283 .415	.131 .434 .577	.226 .570 .718	.300 .695 .803	.395 .782 .886	.470 .841 .915	.640 .922 .962	
$\lambda_3^*$	.052 .190 .323	.121 .365 .553	.238 .559 .748	.364 .707 .859	.509 .834 .929	.622 .906 .974	.719 .934 .983	.864 .987 .998	
$\Gamma_3$	.043 .174 .315	.021 .112 .237	.004 .036 .091	.0 .007 .024	.0 .001 .004	.0 .0 .0	.0 .0 .0	.0 .0 .0	.0

Table 6.2 shows that for  $n = 10$  and the distributions considered, the rank statistics  $W_1$  are consistently less powerful than the corresponding studentized linear spacings statistics  $S_1^*$ . This led us to compute Table 6.3 in which  $n = 30$  and the Weibull distribution is considered. This table shows that for small  $\theta$ , there is no difference in the powers between the total time on test statistic  $S_1^*$  and its rank counter part  $W_1$ , while for larger  $\theta$ , the total time on test statistic is again better. However, as expected from the asymptotic theory, the difference in power seems to be decreasing.

$\alpha$	.01 .05 .10	.01 .05 .10	.01 .05 .10	.01 .05 .10	.01 .05 .10	.01 .05 .10
e	1.10	1.25	1.40	1.75	2.00	2.50
w <sub>I</sub>	.056 .161 .262	.180 .386 .552	.365 .642 .783	.797 .942 .977	.916 .986 .999	.973 .992 .999
s <sub>I</sub> *	.045 .105 .268	.195 .416 .565	.441 .706 .820	.923 .985 .995	.995 1.00 1.00	1.00 1.00 1.00

7. Appendix

We use the notation of Section 4. Heuristically our argument is very simple and is essentially the same as that used in [3].

$$\begin{aligned}
 T_n &= n^{-1/2} \left\{ \sum_{i=1}^n [h(X_{(i)}) - E[h(X_{(i)})]] \right\} \\
 &\doteq n^{-1/2} \left\{ \sum_{i=1}^n h(X_{(i)}) - h(E(X_{(i)})) \right\} \\
 &\doteq n^{-1/2} \left\{ \sum_{i=1}^n h'(E(X_{(i)})) [X_{(i)} - E(X_{(i)})] \right\} \\
 &\doteq n^{-1/2} \sum_{i=1}^n h' \left( -\log \left( 1 - \frac{i}{n+1} \right) \right) \sum_{j=1}^i (D_j - 1)(n - j + 1)^{-1}
 \end{aligned} \tag{7.1}$$

under  $H_0$ . From the last approximate identity our result follows. The justification of these approximations poses some minor technical difficulties. We proceed with the proof of the theorem. We show that i, ii and iii b) suffice. Let,

$$\begin{aligned}
 J_\delta(t) &= 1 & \delta \leq t \leq 1 - \delta \\
 &= 0 & \text{otherwise}
 \end{aligned} \tag{7.2}$$

From Theorem 3 of [3] it follows that, if  $T_n^\delta = n^{-1/2} \sum_{i=1}^n J_\delta \left( \frac{i}{n+1} \right) h(X_{(i)})$  then,

$$T_n^\delta = n^{1/2} \mu_n^\delta + n^{-1/2} \sum_{j=1}^n a_j (j(n+1)^{-1})(D_j - 1) + o_p(1) \tag{7.3}$$

where

$$a_\delta(s) = (1-s)^{-1} \int_s^1 J_\delta(t) h'(-\log(1-t)) dt \tag{7.4}$$

$$\mu_n^\delta = n^{-1} \sum_{i=1}^n J_\delta\left(\frac{i}{n+1}\right) h(-\log(1 - i(n+1)^{-1})) \quad (7.5)$$

and  $o_p(1)$  as usual denotes a remainder converging to 0 in probability.

Now let

$$\gamma_n^\delta = -n^{-1/2} \sum_{j=1}^n J_\delta\left(\frac{j}{n+1}\right) E(h(X_{(j)})) \quad (7.6)$$

$$R_{n1}^\delta = n^{-1/2} \sum_{j < (n+1)\delta} \{h(X_{(j)}) - E(h(X_{(j)}))\} \quad (7.7)$$

and

$$R_{n2}^\delta = n^{-1/2} \sum_{j > (n+1)(1-\delta)} \{h(X_{(j)}) - E(h(X_{(j)}))\} \quad (7.8)$$

Then,

$$T_n = T_n^\delta + R_{n1}^\delta + R_{n2}^\delta + \gamma_n^\delta. \quad (7.9)$$

We begin with,

Lemma 7.1:

If  $\delta_n \rightarrow 0$  as  $n \rightarrow \infty$

$$n^{-1/2} \sum_{i=1}^n \left[ a_{\delta_n}\left(\frac{i}{n+1}\right) - a\left(\frac{i}{n+1}\right) \right] (D_i - 1) \rightarrow 0 \quad (7.10)$$

in probability.

Proof:

$$E \left\{ n^{-1/2} \sum_{i=1}^n \left[ a_{\delta_n}\left(\frac{i}{n+1}\right) - a\left(\frac{i}{n+1}\right) \right] (D_i - 1) \right\}^2 \quad (7.11)$$

$$= n^{-1} \sum_{i=1}^n \left[ a_{\delta_n}\left(\frac{i}{n+1}\right) - a\left(\frac{i}{n+1}\right) \right]^2$$



$$| \alpha_\delta(s) |^2 \leq (1-s)^{-2} \left[ \int_s^1 | h'(-\log(1-t)) | dt \right]^2 \quad \text{for } \delta \geq 0. \quad (7.12)$$

Moreover,

$$\int_0^1 (1-s)^{-2} \left( \int_s^1 | h'(-\log(1-t)) | dt \right)^2 ds < \infty. \quad (7.13)$$

To see (7.13) note that the left-hand side equals

$$\begin{aligned} & \int_0^1 (1-s)^{-2} \int_s^1 | h'(-\log(1-t)) | dt \int_s^1 | h'(-\log(1-v)) | dv ds \\ &= 2 \int_0^\infty \int_0^u | h'(u) | | h'(r) | e^{-u}(1-e^{-r}) dr du \end{aligned} \quad (7.14)$$

after some standard arguments. Now by c) there exists  $\delta > 0$  such that

$h'(x)$  has constant sign for  $x < \delta$ ,  $x > \delta^{-1}$  and  $\sup_{\delta \leq x \leq \delta^{-1}} | h'(x) | < \infty$ .

Then

$$\int_\delta^{\delta^{-1}} \int_\delta^u | h'(u) | | h'(r) | e^{-u}(1-e^{-r}) dr du < \infty.$$

Suppose for simplicity  $h'(x) \geq 0$   $x < \delta$ ,  $h'(x) \leq 0$   $x > \delta^{-1}$ . Then

$$\begin{aligned}
& \int_{\delta}^{\delta^{-1}} \int_0^{\delta} |h'(u)| |h'(r)| e^{-u}(1 - e^{-r}) du dr \\
&= \int_{\delta}^{\delta^{-1}} |h'(u)| e^{-u} \int_0^{\delta} h'(r)(1 - e^{-r}) dr du \\
&= \int_{\delta}^{\delta^{-1}} |h'(u)| \left\{ \int_0^{\delta} h(r) e^{-r} dr + h(\delta)(1 - e^{-\delta}) \right\} du < \infty \\
&\text{since } \int_0^{\infty} e^{-r} |h(r)| dr < \infty.
\end{aligned} \tag{7.15}$$

Continuing,

$$\begin{aligned}
& 2 \int_0^{\delta} \int_0^u |h'(u)| |h'(r)| e^{-u}(1 - e^{-r}) du dr \\
&= \lim_{\lambda \rightarrow 0} 2 \int_{\lambda}^{\delta} h'(u) e^{-u} \{h(u) - h(\lambda)\} du - \int_{\lambda}^{\delta} \frac{\left[ \int_{\lambda}^u e^{-v} h'(v) dv \right]^2}{du} du \\
&= \lim_0 \int_{\lambda}^{\delta} e^{-u} dh^2(u) - 2h(\lambda) \int_{\lambda}^{\delta} h'(u) e^{-u} du - \left[ \int_{\lambda}^{\delta} h'(u) e^{-u} du \right]^2 \\
&= \int_0^{\delta} h^2(u) e^{-u} du - h^2(\delta) e^{-\delta} - 2 \left[ \int_0^{\delta} e^{-u} h(u) du \right] h(\delta) e^{-\delta}
\end{aligned} \tag{7.16}$$

$$+ \lim_{\lambda \rightarrow 0} - \left\{ h^2(\lambda)(1 - e^{-\lambda}) + 2h(\lambda)(1 - e^{-\lambda}) \int_{\lambda}^{\delta} h(u) e^{-u} du + \right. \\ \left. + 2h(\delta) e^{-\delta} h(\lambda)(1 - e^{-\lambda}) \right\} < \infty$$

as long as  $h^2(\lambda)(1 - e^{-\lambda})$  is bounded as  $\lambda \rightarrow 0$ . But this, of course, holds if

$$\int_0^{\epsilon} h^2(u) e^{-u} du < \infty, \text{ and } h \uparrow \text{ in } (0, \delta).$$

One can dispose of the other pieces of the integral by similar arguments.

From (7.11) and (7.12) it readily follows that

$$a_{\delta_n}(s) \rightarrow a(s) \quad (7.17)$$

for every  $0 < s < 1$  and again by (7.12), (7.16) and the dominated convergence theorem the lemma follows. Suppose we can show

$$\lim_{\delta \rightarrow 0} \sup_n \text{Var } R_{ni}^{\delta} = 0 \quad (7.18)$$

for  $i = 1, 2$ . We claim the theorem follows under our assumptions. To see this note that (7.9) and (7.18) imply that for every  $\epsilon > 0$ , there exists a  $\delta$  such that

$$\limsup_n d(T_n, T_n^{\delta} + Y_n^{\delta}) \leq \epsilon \quad (7.19)$$

where

$$d(X, Y) = E \{ |X - Y| (1 + |X - Y|)^{-1} \} \quad (7.20)$$

is the usual metric for convergence in probability. Now, (7.3) yields for  $\delta$  as

above,

$$\limsup_n d\left(T_n, n^{-\frac{1}{2}} \sum_{i=1}^n a_\delta(i(n+1)^{-1})(D_i - 1) + n^{\frac{1}{2}} \mu_n^\delta + \gamma_n^\delta\right) \leq \epsilon \quad (7.21)$$

and by Lemma 7.1, for  $\delta$  sufficiently small.

$$\limsup_n d\left(T_n, S_n + n^{\frac{1}{2}} \mu_n^\delta + \gamma_n^\delta\right) \leq \epsilon \quad (7.22)$$

This, of course, implies that there exists a sequence of constants  $K_n$  such that,

$$\lim_n d(T_n, S_n + K_n) = 0. \quad (7.23)$$

Since  $S_n$  and  $T_n$  are both asymptotically normal with mean 0 by Lemma 4.1 of [2] and (ii) and the central limit theorem, it follows that  $K_n \rightarrow 0$ . We prove (7.18) for  $i = 1$ ;  $i = 2$  is proved analogously. Let  $\delta$  be defined as in the discussion preceding (7.15). Define,

$$\begin{aligned} Z_1 &= h(X_1) \quad \text{if } 0 \leq X_1 < -\log(1 - \delta) \\ &= U_1 \quad \text{otherwise} \end{aligned} \quad (7.24)$$

where the  $U_1$  are uniformly distributed on  $(-\log(1 - \delta), B)$  and independent of each other and the  $X_1$ .  $B$  is so chosen that the density of  $Z_1$  at  $h(-\log(1 - \delta))$  is the same as the density of  $h(X_1)$  at  $h(-\log(1 - \delta))$ .

We argue as in [2]. Let  $Z_{(1)} < \dots < Z_{(n)}$  be the order statistics of the  $Z_i$ 's. Then,

$$\begin{aligned}
& E \left( \sum_{i \leq \frac{\delta n}{2}} (h(X_{(i)}) - Z_{(i)}) \right)^2 \\
&= E \left( \sum_{i \leq \frac{\delta n}{2}} (h(X_{(i)}) - Z_{(i)}) I_{[X_{(i)} \geq -\log(1-\delta)]} \right)^2 \\
&\leq 2 E \left( \sum_{i \leq \frac{\delta n}{2}} Z_{(i)} I_{[X_{(i)} \geq -\log(1-\delta)]} \right)^2 \\
&\quad + 2 \sum_{i \leq \frac{\delta n}{2}} \left[ E \left( h^2(X_{(i)}) I_{[X_{(i)} \geq -\log(1-\delta)]} \right) \right]^{\frac{1}{2}}.
\end{aligned} \tag{7.25}$$

The first identity follows by definition of the  $Z_i$ , the second inequality from the  $c_r$  and Minkowski inequalities. Now,

$$E \left( \sum_{i \leq \frac{\delta n}{2}} Z_{(i)} I_{[X_{(i)} \geq -\log(1-\delta)]} \right)^2 \leq B^2 \frac{\delta^2 n^2}{4} P \left[ X_{\left[ \frac{\delta n}{2} \right]} \geq -\log(1-\delta) \right] \rightarrow 0 \tag{7.26}$$

since by [2] Lemma 2.2,  $P \left[ X_{\left[ \frac{\delta n}{2} \right]} \geq -\log(1-\delta) \right] \rightarrow 0$  exponentially.

On the other hand,

$$\begin{aligned}
& E \left( h^2(X_{(i)}) I_{[X_{(i)} \geq -\log(1-\delta)]} \right) \\
&= \int_{v \geq -\log(1-\delta)} h^2(v) n! [(i-1)! (n-i)!]^{-1} e^{-(n-i+1)v} \\
&\quad (1 - e^{-v})^{i-1} dv \\
&\leq \sup_{v \geq -\log(1-\delta)} e^{-(n-1)v} (1 - e^{-v})^{i-1} n \binom{n-1}{i-1} E(h^2(X_1)).
\end{aligned} \tag{7.27}$$

Now,

$$\sup_{v \geq -\log(1-\delta)} e^{-(n-1)v} (1 - e^{-v})^{i-1} n \binom{n-1}{i-1} \quad (7.28)$$

$$= \delta^{i-1} (1 - \delta)^{n-i} n \binom{n-1}{i-1} \quad \text{for } i \leq \frac{\delta}{2} n.$$

Finally,

$$\sup_{i \leq \frac{\delta n}{2}} \delta^{i-1} (1 - \delta)^{n-i} n \binom{n-1}{i-1} \quad (7.29)$$

$$= \delta^{\left[\frac{\delta n}{2}\right]-1} (1 - \delta)^{n - \left[\frac{\delta n}{2}\right]} n \binom{n-1}{\left[\frac{\delta n}{2}\right]-1}$$

by an easy induction argument on  $i$ . By Lemma 2.2 of [2] the right-hand side of (7.29)  $\rightarrow 0$  exponentially. Then, (7.27), (7.28) and (7.29) imply that

$\lim_{\delta \rightarrow 0} \limsup_n \text{Var } R_{n1}^{\delta} = 0$  if and only if

$$\lim_{\delta \rightarrow 0} \limsup_n \text{Var} \left[ \sum_{i \leq \delta' n} Z_{(i)} \right] = 0 \quad (7.30)$$

where  $Z_i$  are defined for fixed  $\delta$ . Since the  $Z_i$ 's are independently and identically distributed with a density positive on its convex support and  $E(Z_1^2) < \infty$ , (7.30) follows from [2]. The sufficiency of (i), (ii) and (iii) b) for the conclusion of the theorem follows. The sufficiency of (i), (ii), and (iii) a) is an easy consequence of theorem of [3].

Using the methods of [2] on  $T_n^\delta$  one can show that  $E(S_n - T_n)^2 \rightarrow 0$ .

As in [3] the smoothness conditions on  $h$  may be relaxed.

## REFERENCES

- [1] Barlow, R. (1967). Likelihood ratio tests for restricted families. *Technical Report* ORC 67-16.
- [2] Bickel, P. J. (1967). Some contributions to the theory of order statistics. *Proc. Vth Berk. Symp.*, VI. University of California Press.
- [3] Chernoff, H., Gastwirth, J. and Johns, M. V. (1967). Asymptotic distribution of linear combinations of functions of order statistics with application to estimation. *Ann. Math. Statist.* 38 52-73.
- [4] Hajek, J. and Sidak, Z. (1967). *Theory of Rank Tests*. Academic Press, New York.
- [5] Hajek, J. (1961). Some extensions of the Wald-Wolfowitz-Noether theorems. *Ann. Math. Statist.* 32 506-523.
- [6] Hodges, J. L. and Lehmann, E. L. (1961). Comparison of the normal scores and Wilcoxon tests. *Proc. IVth Berk. Symp.* University of California Press.
- [7] Hoeffding, W. (1951). "Optimum" nonparametric tests. *Proc. IInd Berk. Symp.* University of California Press.
- [8] LeCam, L. (1966). Likelihood functions for large numbers of independent observations. *Festschrift for J. Neyman*. J. Wiley and Sons, New York.
- [9] Lehmann, E. L. (1966). Some concepts of dependence. *Ann Math. Statist.* 37 1137-1154.
- [10] Proschan, F. and Pyke R. (1967). Tests for monotone failure rate. *Proc. Vth Berk. Symp.*, VIII. University of California Press.
- [11] van Zwet, W. (1964). *Convex Transformations of Random Variables*. Math Centrum, Amsterdam.
- [12] Wald, A. (1941). Asymptotically most powerful tests of statistical hypotheses. *Ann. Math. Statist.* 12 1-19.
- [13] Barlow R. E. and Proschan F. (1966). Inequalities for linear combinations of order statistics from restricted families. *Ann. Math. Statist.* 37 1574-1591.
- [14] Nadler J. and Eilbott J. (1967). Testing for monotone failure rates. (Unpublished).

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